

The integral cohomology of configuration spaces of pairs of points in real projective spaces

Carlos Domínguez*, Jesús González†, and Peter Landweber

Abstract

We compute the integral cohomology ring of configuration spaces of two points on a given real projective space. Apart from an integral class, the resulting ring is a quotient of the known integral cohomology of the dihedral group of order 8 (in the case of unordered configurations, thus has only 2- and 4-torsion) or of the elementary abelian 2-group of rank 2 (in the case of ordered configurations, thus has only 2-torsion). As an application, we complete the computation of the symmetric topological complexity of real projective spaces $P^{2^i+\delta}$ with $i \geq 0$ and $0 \leq \delta \leq 2$.

Key words and phrases: 2-point configurations of real projective spaces; dihedral group of order 8; Bockstein spectral sequence; symmetric topological complexity; Euclidean embedding dimension.

2010 Mathematics Subject Classification: Primary: 55R80, 55T10; Secondary: 55M30, 57R19, 57R40.

1 A brief outline of the paper

We compute the integral cohomology rings of $F(P^m, 2)$ and $B(P^m, 2)$, the configuration spaces of two distinct points, ordered and unordered respectively, in the m -dimensional real projective space P^m . Our explicit results are presented in Theorems 2.1–2.3 for $F(P^m, 2)$, and in Theorems 2.6–2.8 for $B(P^m, 2)$. Proofs are given in Section 4 for $F(P^m, 2)$, and in Sections 5 and 6 for $B(P^m, 2)$.

These rather technical calculations arose from a study of the symmetric topological complexity (TC^S) of P^m , and its relation to the embedding dimension of this manifold (Section 3 recalls the basics of this relationship). In particular, our cohomological calculations allow us to complete the determination, started in [11], of $TC^S(P^{2^i+\delta})$ for $i \geq 0$ and $0 \leq \delta \leq 2$. The explicit new TC^S -result is given in Theorem 3.1; the global TC^S -picture for these projective spaces is summarized in (20)–(22).

*Supported by Conacyt Ph.D. scholarship number 162645.

†Partially supported by CONACYT Research Grant number 102783.

2 Cohomology rings

Unless indicated otherwise, the notation $H^*(X)$ refers to the integral cohomology ring of a space X where a simple system of local coefficients is used. The degree of a cohomology class is explicitly indicated by means of an subscript: $c_k \in H^k(X)$. The cyclic group with 2^e elements is denoted by \mathbb{Z}_{2^e} . In the case $e = 1$ we also use the notation \mathbb{F}_2 if the field structure is to be noted. It will be convenient to use the notation $\langle k \rangle$ for the elementary abelian 2-group of rank k , and write $\{k\}$ as a shorthand for $\langle k \rangle \oplus \mathbb{Z}_4$.

Recall that the ring $H^*(P^\infty \times P^\infty)$ is generated over the integers by three classes x_2, y_2 , and z_3 subject only to the four relations

$$2x_2 = 0, \quad 2y_2 = 0, \quad 2z_3 = 0, \quad \text{and} \quad z_3^2 + x_2y_2(x_2 + y_2) = 0. \quad (1)$$

The mod 2 reduction map $\rho: H^*(P^\infty \times P^\infty) \rightarrow H^*(P^\infty \times P^\infty; \mathbb{F}_2)$ is characterized by

$$\rho(x_2) = x_1^2, \quad \rho(y_2) = y_1^2, \quad \text{and} \quad \rho(z_3) = x_1y_1(x_1 + y_1). \quad (2)$$

Here $x_1, y_1 \in H^*(P^\infty \times P^\infty; \mathbb{F}_2) = H^*(P^\infty; \mathbb{F}_2) \otimes H^*(P^\infty; \mathbb{F}_2)$ are given by $x_1 = z_1 \otimes 1$ and $y_1 = 1 \otimes z_1$ where $z_1 \in H^1(P^\infty; \mathbb{F}_2)$ is the generator (cf. [16, Example 3E.5]). We also use the notation x_2, y_2 , and z_3 (with integral coefficients), as well as x_1 and y_1 (with mod 2 coefficients) for the images of the corresponding classes under the homomorphism of cohomology rings induced by the obvious inclusion

$$\alpha: F(P^m, 2) \hookrightarrow P^\infty \times P^\infty. \quad (3)$$

Theorem 2.1. *Let $m = 2t + \delta$, $\delta \in \{0, 1\}$. The following relations hold in $H^*(F(P^m, 2))$:*

$$x_2^{t+1} = 0, \quad y_2^{t+1} = 0, \quad \text{and} \quad \sum_{i,j \geq 0, i+j=t} x_2^i y_2^j z_3 = 0. \quad (4)$$

- (a) *If $\delta = 0$, the integral cohomology ring $H^*(F(P^m, 2))$ is generated by x_2, y_2, z_3 , and a class w_{2m-1} subject only to the relations (1), (4), and*

$$x_2^t y_2^t = 0, \quad \sum x_2^i y_2^j z_3 = 0, \quad \text{and} \quad w_{2m-1} \mu = 0, \quad (5)$$

for $\mu \in \{x_2, y_2, z_3, w_{2m-1}\}$, where the sum in (5) runs over $i, j \geq 0$ with $i + j = t - 1$.

- (b) *If $\delta = 1$, the integral cohomology ring $H^*(F(P^m, 2))$ is generated by x_2, y_2, z_3 , and a class w_m subject only to the relations (1), (4), and*

$$w_m y_2 + x_2^t z_3 = 0 \quad \text{and} \quad w_m \mu = 0, \quad \text{for } \mu \in \{x_2, z_3, w_m\}. \quad (6)$$

Note that the $(x_2 \text{ vs. } y_2)$ -symmetry in the presentation for $H^*(F(P^{2t}, 2))$ no longer holds in (6). Although this is an intrinsic phenomenon for $m \equiv 3 \pmod{4}$, the asymmetry is only apparent for $m \equiv 1 \pmod{4}$: in terms of the torsion-free generator $w'_{4\ell+1} = w_{4\ell+1} + z_3(x_2^{2\ell-1} + x_2^{2\ell-2}y_2 + \cdots + x_2^\ell y_2^{\ell-1})$, (6) is replaced by the $(x_2 \text{ vs. } y_2)$ -symmetric relations $w'_{4\ell+1}x_2 = (x_2^{2\ell} + \cdots + x_2^{\ell+1}y_2^{\ell-1})z_3$, $w'_{4\ell+1}y_2 = (y_2^{2\ell} + \cdots + y_2^{\ell+1}x_2^{\ell-1})z_3$, $w'_{4\ell+1}z_3 = x_2^{\ell+1}y_2^{\ell+1}$, and $(w'_{4\ell+1})^2 = 0$.

The relations listed in Theorem 2.1 are minimal for $m \geq 3$, and lead to explicit descriptions of cohomology groups (Theorem 2.2 next) and \mathbb{F}_2 -bases for torsion subgroups (Theorem 2.3 following).

Theorem 2.2. *For $t \geq 1$,*

$$H^i(F(P^{2t}, 2)) = \begin{cases} \mathbb{Z}, & i = 0 \text{ or } i = 4t - 1; \\ \langle \frac{i}{2} + 1 \rangle, & i \text{ even}, 1 \leq i \leq 2t; \\ \langle \frac{i-1}{2} \rangle, & i \text{ odd}, 1 \leq i \leq 2t; \\ \langle 2t + 1 - \frac{i}{2} \rangle, & i \text{ even}, 2t < i < 4t - 1; \\ \langle 2t - \frac{i+1}{2} \rangle, & i \text{ odd}, 2t < i < 4t - 1; \\ 0, & \text{otherwise.} \end{cases}$$

For $t \geq 0$,

$$H^i(F(P^{2t+1}, 2)) = \begin{cases} \mathbb{Z}, & i = 0; \\ \langle \frac{i}{2} + 1 \rangle, & i \text{ even}, 1 \leq i \leq 2t; \\ \langle \frac{i-1}{2} \rangle, & i \text{ odd}, 1 \leq i \leq 2t; \\ \mathbb{Z} \oplus \langle t \rangle, & i = 2t + 1; \\ \langle 2t + 1 - \frac{i}{2} \rangle, & i \text{ even}, 2t + 1 < i \leq 4t + 1; \\ \langle 2t + 1 - \frac{i-1}{2} \rangle, & i \text{ odd}, 2t + 1 < i \leq 4t + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let $m = 2t + \delta$ with $\delta \in \{0, 1\}$. A graded \mathbb{F}_2 -basis for the torsion subgroups of $H^*(F(P^m, 2))$ can be chosen as follows: In even dimensions the basis consists of the monomials $x_2^i y_2^j$ with $0 \leq i, j \leq t$, $(i, j) \neq (0, 0)$ and, if $\delta = 0$, $(i, j) \neq (t, t)$. In odd dimensions the basis consists of monomials $x_2^i y_2^j z_3$ with $0 \leq i \leq t - 1 + \delta$ and $0 \leq j \leq t - 2 + \delta$.*

The following is a straightforward consequence of the three results above.

Corollary 2.4. *The map induced in integral cohomology by (3):*

1. *surjects in positive dimensions onto the torsion subgroups of $H^*(F(P^m, 2))$;*
2. *has cokernel generated by w_m (when m is odd) and w_{2m-1} (when m is even);*
3. *is injective in dimensions at most m .*

Corollary 2.4.3 can be stated in more precise terms: $\text{Ker}(\alpha^*)$ is the ideal of $H^*(P^\infty \times P^\infty)$ generated by the right-hand-side terms of the equations in (4)–(6). Earlier versions of this paper (available as [12]) interpret the latter fact in terms of Fadell-Husseini's index theory. The proofs of Theorems 2.1–2.3 rely on first establishing the first two assertions of Corollary 2.4 through a Bockstein spectral sequence argument.

Next we focus on $B(P^m, 2)$. Recall the following three facts about the dihedral group D_8 of order 8 (see for instance [14]). The ring $H^*(D_8)$ is generated over the integers by four classes a_2, b_2, c_3 , and d_4 subject only to the six relations

$$2a_2 = 0, \quad 2b_2 = 0, \quad 2c_3 = 0, \quad 4d_3 = 0, \quad b_2^2 + a_2b_2 = 0, \quad \text{and} \quad c_3^2 + a_2d_4 = 0. \quad (7)$$

The \mathbb{F}_2 -algebra $H^*(D_8; \mathbb{F}_2)$ is generated by three classes u_1, v_1, w_2 subject only to

$$u_1^2 = u_1v_1. \quad (8)$$

The mod 2 reduction map $\rho: H^*(D_8) \rightarrow H^*(D_8; \mathbb{F}_2)$ is characterized by

$$\rho(a_2) = v_1^2, \quad \rho(b_2) = u_1v_1, \quad \rho(c_3) = v_1w_2, \quad \text{and} \quad \rho(d_4) = w_2^2. \quad (9)$$

We also use the notation a_2, b_2, c_3 , and d_4 (with integral coefficients), as well as u_1, v_1 , and w_2 (with mod 2 coefficients) for the images of the corresponding classes under the map

$$\beta: B(P^m, 2) \rightarrow BD_8 \quad (10)$$

that classifies the following action (cf. [13, Proposition 2.6]):

Definition 2.5. In the usual wreath product extension $1 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow D_8 \rightarrow \mathbb{Z}_2 \rightarrow 1$, let $\rho_1, \rho_2 \in D_8$ be the obvious generators of the normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$, and let (the class of) $\rho \in D_8$ generate the quotient group \mathbb{Z}_2 so that, via conjugation, ρ switches ρ_1 and ρ_2 . D_8 acts freely on the Stiefel manifold $V_{m+1,2}$ of orthonormal 2-frames in \mathbb{R}^{m+1} by setting $\rho(v_1, v_2) = (v_2, v_1)$, $\rho_1(v_1, v_2) = (-v_1, v_2)$ and $\rho_2(v_1, v_2) = (v_1, -v_2)$, so that the orbit space $V_{m+1,2}/D_8$ is contained in $B(P^m, 2)$ as a strong deformation retract.

Theorem 2.6. *Let $m = 2t + \delta$, $\delta \in \{0, 1\}$ and, for $r \geq 0$, consider the elements*

$$\sigma_{2r} = \sum_{\substack{i, j \geq 0 \\ i + 2j = r}} \binom{i+j}{j} a_2^i d_4^j \quad \text{and} \quad \iota_{2r} = \begin{cases} 2d_4^{\frac{r}{2}}, & \text{if } r \text{ is even;} \\ 0, & \text{if } r \text{ is odd;} \end{cases}$$

in $H^(B(P^m, 2))$. The following relations hold in $H^*(B(P^m, 2))$:*

$$a_2\sigma_{2t} = 0, \quad b_2\sigma_{2t} + \iota_{2t+2} = 0, \quad \text{and} \quad c_3\sigma_{2t} = 0. \quad (11)$$

- (a) If $\delta = 0$, the integral cohomology ring $H^*(B(P^m, 2))$ is generated by a_2, b_2, c_3, d_4 , and a class e_{2m-1} subject only to the relations (7), (11), and

$$c_3\sigma_{2t-2} = 0, \quad b_2d_4\sigma_{2t-2} + \iota_{2t+4} = 0, \quad d_4^t = 0, \quad \text{and} \quad e_{2m-1}\mu = 0, \quad (12)$$

for $\mu \in \{a_2, b_2, c_3, d_4, e_{2m-1}\}$.

- (b) If $\delta = 1$, the integral cohomology ring $H^*(B(P^m, 2))$ is generated by a_2, b_2, c_3, d_4 , and a class e_m subject only to the relations (7), (11),

$$a_2\sigma_{2t+2} = 0, \quad b_2\sigma_{2t+2} + \iota_{2t+4} = 0, \quad c_3\sigma_{2t+2} = 0, \quad d_4^{t+1} = 0, \quad (13)$$

$$e_m^2 = 0, \quad \mu e_m = \kappa b_2^\kappa c_3 d_4^\ell, \quad c_3 e_m = \eta d_4^{\ell+1}, \quad \text{and} \quad d_4 e_m = \sum_{i=1}^{\ell} \binom{t-i}{i-1} a_2^{t-2i} b_2 c_3 d_4^i. \quad (14)$$

Here $\mu \in \{a_2, b_2\}$, $t = 2\ell + \kappa$ with $\kappa \in \{0, 1\}$, and $\eta = b_2$ if $\kappa = 1$, whereas $\eta = 2$ if $\kappa = 0$, except perhaps for $m = 5$.

For $m = 5$, it is natural to expect $\eta = 2$ in the product $c_3 e_5$ appearing in (14). Our methods assure, in any case, $\eta \in \{0, 2\}$. For $m = 3$, the third relation in (14) gives $c_3 e_3 = b_2 d_4$, a trivial element in view of Theorem 2.7 below—more explicitly, one can use Lemma 6.2.1 in the final section of the paper. Except for the latter situation, the right hand side of each relation in (14) is in ‘reduced’ form, as follows from Theorem 2.8 below. In fact, the relations listed in Theorem 2.6 are minimal for $m \geq 3$, and lead to explicit descriptions of cohomology groups (Theorem 2.7 next) and minimal generators for torsion subgroups (Theorem 2.8 following).

Theorem 2.7. *Let $0 \leq b \leq 3$. For $t \geq 1$,*

$$H^{4a+b}(B(P^{2t}, 2)) = \begin{cases} \mathbb{Z}, & 4a + b = 0 \text{ or } 4a + b = 4t - 1; \\ \{2a\}, & b = 0 < a, \ 4a + b \leq 2t; \\ \langle 2a \rangle, & b = 1, \ 4a + b \leq 2t; \\ \langle 2a + 2 \rangle, & b = 2, \ 4a + b \leq 2t; \\ \langle 2a + 1 \rangle, & b = 3, \ 4a + b \leq 2t; \\ \{2t - 2a\}, & b = 0, \ 2t < 4a + b < 4t - 1; \\ \langle 2t - 2a - 1 \rangle, & b = 1, \ 2t < 4a + b < 4t - 1; \\ \langle 2t - 2a \rangle, & b = 2, \ 2t < 4a + b < 4t - 1; \\ \langle 2t - 2a - 2 \rangle, & b = 3, \ 2t < 4a + b < 4t - 1; \\ 0, & \text{otherwise.} \end{cases}$$

For $t \geq 0$,

$$H^{4a+b}(B(\mathbb{P}^{2t+1}, 2)) = \begin{cases} \mathbb{Z}, & 4a + b = 0; \\ \{2a\}, & b = 0 < a, \ 4a + b \leq 2t; \\ \langle 2a \rangle, & b = 1, \ 4a + b \leq 2t; \\ \langle 2a + 2 \rangle, & b = 2, \ 4a + b \leq 2t; \\ \langle 2a + 1 \rangle, & b = 3, \ 4a + b \leq 2t; \\ \mathbb{Z} \oplus \langle t \rangle, & 4a + b = 2t + 1; \\ \{2t - 2a\}, & b = 0, \ 2t + 1 < 4a + b \leq 4t + 1; \\ \langle 2t - 2a + 1 \rangle, & b = 1, \ 2t + 1 < 4a + b \leq 4t + 1; \\ \langle 2t - 2a \rangle, & b \in \{2, 3\}, \ 2t + 1 < 4a + b \leq 4t + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.8. *Let $m = 2t + \delta$ with $\delta \in \{0, 1\}$. A minimal set of generators for the torsion subgroups of $H^*(B(\mathbb{P}^m, 2))$ is given by the monomials*

$$a_2^i b_2^\varepsilon d_4^j \text{ (in even dimensions) and } a_2^i b_2^\varepsilon c_3 d_4^j \text{ (in odd dimensions)} \quad (15)$$

where $\varepsilon \in \{0, 1\}$, $i, j \geq 0$, $j \leq t + \delta - 1$, and

- $1 \leq i + j + \varepsilon \leq t$ in even dimensions;
- $i + j + 1 < t + \delta$ in odd dimensions (note that this condition is independent of ε).

The following is a straightforward consequence of the last three results.

Corollary 2.9. *The map induced in integral cohomology by (10):*

1. *surjects in positive dimensions onto the torsion subgroups of $H^*(B(\mathbb{P}^m, 2))$;*
2. *has cokernel generated by e_m (when m is odd) and e_{2m-1} (when m is even);*
3. *is injective in dimensions at most m .*

Note that $B(\mathbb{P}^m, 2)$ and $F(\mathbb{P}^m, 2)$ become homology spheres after inverting 2. Such a fact holds integrally in the case of $B(\mathbb{P}^1, 2)$ and $F(\mathbb{P}^1, 2)$. Indeed, there are well-known homotopy equivalences

$$F(\mathbb{P}^1, 2) \simeq S^1 \simeq B(\mathbb{P}^1, 2) \quad (16)$$

(cf. [18, Example 2.2]). Since our descriptions of the integral cohomologies of $F(\mathbb{P}^1, 2)$ and $B(\mathbb{P}^1, 2)$ are compatible with (16), we will assume $m > 1$ in Sections 4–6. The case of \mathbb{P}^2 is the only further situation where the ring structure is trivial integrally—with $z_3 = 0$ for $H^*(F(\mathbb{P}^2, 2))$, and $c_3 = d_4 = 0$ for $H^*(B(\mathbb{P}^2, 2))$.

Our results can be coupled with the Universal Coefficient Theorem, expressing homology in terms of cohomology, to give an explicit description of the integral homology groups

of $F(\mathbb{P}^m, 2)$ and $B(\mathbb{P}^m, 2)$. Likewise, in combination with Poincaré duality (in its not necessarily orientable version, cf. [16, Theorem 3H.6] or [23, Theorem 4.51]), our results lead to explicit descriptions of the w_1 -twisted homology and cohomology groups of $F(\mathbb{P}^m, 2)$ and $B(\mathbb{P}^m, 2)$. Details are given in [12].

Theorems 2.6–2.8 fully extend the calculations of $H^i(B(\mathbb{P}^m, 2))$ given in [1] for i close to the top cohomological dimension $2m - 1$. Bausum’s work led to a description of the sets of isotopy classes of smooth embeddings of \mathbb{P}^m in \mathbb{R}^{2m-e} for low values of e (as low as $e \leq 2$). Similar results were obtained by Larmore and Rigdon (note the implicit hypothesis $m > 3$ in [19, Section 4])¹. Instead, our TC^S -application follows the method outlined in [10].

Our original (additive) approach to $H^*(B(\mathbb{P}^m, 2))$ and $H^*(F(\mathbb{P}^m, 2))$ was based on the Cartan-Leray spectral sequence of the D_8 -action in Definition 2.5, and of the restricted action to the normal subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$. This eventually gave the ring structures (the case of $B(\mathbb{P}^m, 2)$ is part of the Ph.D. thesis of the first author). The Bockstein spectral sequence approach in this paper was suggested by the referee, and leads to condensed proofs—despite that we have spent quite some space giving concrete details and explicit examples of our technical arguments. However, the current gain in brevity sacrifices the geometric motivation in [12], replacing it by a highly technical bookkeeping of cohomology groups through very explicit generators and relations. Thus, it is worth keeping in mind that [4, 12] offer (and make use of) a more geometric understanding of the central role played by D_8 . In particular, [4] explains how the relations (4)–(6) and (11)–(14) arise naturally as key differentials in the relevant Cartan-Leray spectral sequences.

3 Symmetric topological complexity

We now apply the cohomological information in the previous section to the problem of computing the symmetric topological complexity (TC^S) of real projective spaces. As a motivation, we begin with a description of the relationship between TC^S and the embedding dimension of these manifolds. The relevant references for the facts in the next paragraph are [10, 11], and we assume familiarity with the notation in those papers.

Consider the homotopy class

$$B(\mathbb{P}^m, 2) \xrightarrow{u} \mathbb{P}^\infty \tag{17}$$

classifying the obvious double cover $F(\mathbb{P}^m, 2) \rightarrow B(\mathbb{P}^m, 2)$. With the seven possible exceptions² of m explicitly described in [10, Equation (8)], $\text{Emb}(\mathbb{P}^m)$ —the dimension of the smallest Euclidean space in which \mathbb{P}^m can be smoothly embedded—is characterized as the

¹We thank Sadok Kallel for pointing out the results in [1] and [19].

²Remark 3.2 below observes that we can now rule out the first of these potential exceptions.

smallest integer $e(m)$ such that the map in (17) can be homotopy compressed into $P^{e(m)-1}$. On the other hand, the main result in [11] asserts that, without restriction on m , $e(m)$ agrees with Farber-Grant’s symmetric topological complexity³ of P^m , $TC^S(P^m)$. The latter is an invariant proposed in [7] to measure the inherent topological difficulties in the problem of finding “efficient” motion algorithms in robotics. Consequently, potentially new nonembedding results for P^m —as well as inherent difficulties in the problem of planning symmetric motion in P^m —could be deduced from the simple observation that, for a generalized cohomology theory h^* with products, every class $z \in h^*(P^\infty)$ must satisfy

$$u^*(z)^{e(m)} = 0. \quad (18)$$

The idea actually goes back at least as far as [13], where mod 2 coefficients (and obstruction theory) are used. But the \mathbb{Z}_4 groups appearing in Theorem 2.7 carry finer information not yet explored⁴. For instance, the strategy using integral coefficients has recently been exploited in [10] in order to compute $TC^S(SO(3))$ —identifying it as the unique obstruction in Goodwillie’s embedding Taylor tower for P^3 . The same idea now leads to:

Theorem 3.1. $TC^S(P^5) = TC^S(P^6) = 9$.

Before proving this result, we compare it (in Remark 3.2 below) with known information (summarized in [3]) on $\text{Emb}(P^m)$ for $m = 5, 6, 7$, pausing to explain the way Theorem 3.1 gives an exceptional situation to some general patterns of values of $TC^S(P^m)$ (see (20)–(22)).

Remark 3.2. Since $\text{Emb}(P^5) = 9$ ([17, 20]), the list in [10] of seven exceptional values of m for which the equality $\text{Emb}(P^m) = TC^S(P^m)$ *could* fail reduces now to $\{6, 7, 11, 12, 14, 15\}$. Note that 6 is the smallest m for which $\text{Emb}(P^m)$ is unknown: $\text{Emb}(P^6) \in \{9, 10, 11\}$ is the best assertion known to date ([5, 20]). On the other hand, Theorem 3.1 obviously implies $TC^S(P^7) \geq 9$, improving by 1 the previously known best lower bound for $TC^S(P^7)$ noted in [10, Table 1]. In fact, taking into account Rees’ PL embedding $P^7 \subset \mathbb{R}^{10}$ constructed in [25], the above considerations imply that both $TC^S(P^7)$ and $\text{Emb}_{\text{PL}}(P^7)$ lie in $\{9, 10\}$. This contrasts with the best known assertion about the smooth embedding dimension of P^7 , namely $\text{Emb}(P^7) \in \{9, 10, 11, 12\}$ ([15, 21]).

Except for three special cases (related to the Hopf invariant one problem), the reduced version of Farber’s original (non-symmetric) topological complexity captures the immersion dimension of real projective spaces: As proved in [9], the equality $\text{Imm}(P^m) = TC(P^m)$ holds for $m \neq 1, 3, 7$. However, Remark 3.2 suggests that the equality $\text{Emb}(P^m) =$

³We follow the convention in [10] of using the reduced version of TC^S , i.e. we choose to normalize the Schwarz genus of a product fibration $F \times B \rightarrow B$ to be 0—not 1.

⁴Compare with the situation in [2] where the topological Borsuk problem for \mathbb{R}^3 is studied via Fadell-Husseini index theory.

$\mathrm{TC}^S(\mathbb{P}^m)$ could actually hold for every m , at least if Emb is interpreted as *topological* embedding dimension. From such a perspective, it would be highly desirable to know whether \mathbb{P}^6 topologically embeds in \mathbb{R}^9 . On the other hand, it does not seem likely that \mathbb{P}^7 could possibly embed in \mathbb{R}^9 (even topologically), and the techniques proving Theorem 3.1 (using perhaps a cohomology theory better suited than singular cohomology) might allow us to formalize our intuition—we hope to come back to such a point elsewhere.

Before getting into the main technical computation of this section, we set Theorem 3.1 in context. The inequality

$$\mathrm{TC}^S(X) - \mathrm{TC}(X) \geq 0 \quad (19)$$

is proved in [7, Corollary 9] for any space X . It is optimal since, as proved in [11], (19) becomes an equality if X is, for instance, a complex projective space. However, as discussed in [11, Example 3.3], there is no current indication that the left hand side in (19) should even be a bounded function of m for $X = \mathbb{P}^m$. We discuss next the known situation (as updated by Theorem 3.1) for a few particular families of m . We use [3, 9] as the main references for the known numerical values of $\mathrm{TC}(\mathbb{P}^m)$.

To begin with, Example 3.3 in [11] observes that

$$\mathrm{TC}^S(\mathbb{P}^{2^i}) - \mathrm{TC}(\mathbb{P}^{2^i}) = 1 \quad (20)$$

for any $i \geq 0$ (the case $i = 0$ was not mentioned in [11], but it is covered by the calculations in [6, 7]). Example 3.3 in [11] also notes that

$$\mathrm{TC}^S(\mathbb{P}^{2^i+1}) - \mathrm{TC}(\mathbb{P}^{2^i+1}) = 2 \quad (21)$$

for any $i \geq 3$; the corresponding result for $i = 1, 2$ is also true in view of [10] (for $i = 1$) and Theorem 3.1 (for $i = 2$). Lastly, Example 3.3 in [11] remarks that

$$\mathrm{TC}^S(\mathbb{P}^{2^i+2}) - \mathrm{TC}(\mathbb{P}^{2^i+2}) = 1 \quad (22)$$

for any $i \geq 4$. Now, while (22) is also true for $i = 3$ (as remarked in [10, Table 1]), Theorem 3.1 implies that, for $i = 2$, (22) must be replaced by $\mathrm{TC}^S(\mathbb{P}^6) - \mathrm{TC}(\mathbb{P}^6) = 2$.

Returning to this section's main focus (the proof of Theorem 3.1), we take advantage of the obvious inequality $e(m) \leq e(m+1)$ and of the fact that $e(6) \leq 9$ —proved in [24, Corollary 11]—to reduce the proof of Theorem 3.1 to proving the inequality $e(5) \geq 9$. For this purpose, since the plan is to use integral cohomology, it will be simpler to replace (18) by the observation that any cohomology class $z_d \in H^d(\mathbb{P}^\infty)$ with $d \geq e(m)$ must lie in the kernel of u^* . Thus, Theorem 3.1 is a consequence of:

Theorem 3.3. *For $m = 5$, the homomorphism on integral cohomology induced by the map in (17) is monic in dimension 8.*

The proof of Theorem 3.3 is based on Handel's observation (Lemma 3.4 below) that (17) factors through the classifying space of the dihedral group D_8 of order 8.

Lemma 3.4. *The map in (17) corresponds to the pullback under (10) of the class u_1 appearing in (8).*

Proof. This is proved in [13, Proposition 3.5] under the extra hypothesis $m \geq 3$, but the restriction can be removed by naturality. \square

Thus, the homotopy class in (17) factors as $B(P^m, 2) \xrightarrow{\beta} BD_8 \xrightarrow{q} P^\infty$, where q corresponds to the cohomology class $u_1 \in H^1(D_8; \mathbb{F}_2)$. Our last ingredient for the proof of Theorem 3.3 is a description of the effect of q in integral cohomology. With this in mind, we note that the group extension in Definition 2.5 gives a fibration

$$P^\infty \times P^\infty \xrightarrow{\iota} BD_8 \xrightarrow{q'} P^\infty.$$

On the other hand, Handel's proof of [13, Proposition 3.5] characterizes u_1 as the only nonzero element in $H^1(BD_8; \mathbb{F}_2)$ mapping trivially under the fiber inclusion ι . Thus, in fact $q = q'$. In particular, the map induced by q in integral cohomology can be computed in purely algebraic terms, using the projection in the group extension in Definition 2.5. Actually, since $H^*(P^\infty) = \mathbb{Z}[z_2]/2z_2$ where $z_2 \in H^2(P^\infty) = \mathbb{Z}_2$ is the generator, q^* is determined by its value on z_2 . A simple exercise using the Wall-Hamada resolution of the trivial D_8 -module \mathbb{Z} (see for instance [14]) shows that our generators in (7) can be chosen⁵ so that

$$q^*(z_2) = b_2. \tag{23}$$

Proof of Theorem 3.3. In view of (23) and Lemma 3.4 we only need to check that $b_2^4 \neq 0$ in $H^*(B(P^5, 2))$ —a straightforward task in view of our fine cohomological control of $B(P^5, 2)$:

$$\begin{aligned} b_2^4 &= a_2^3 b_2 && \text{in view of the fifth relation in (7)} \\ &= a_2 b_2 d_4 && \text{in view of the second relation in (11)} \\ &= 2d_4^2 && \text{in view of Lemma 6.2.1 with } s = 1. \end{aligned}$$

But d_4^2 is an element of order 4 in view of Theorems 2.7 and 2.8. \square

Remark 3.5. The same method recovers the equation $\text{TC}^S(P^3) = 5$, proved in [10, Theorem 1.4]. It should be noted that the cup-power of $b_2 \in H^*(B(P^m, 2))$ —i.e. the highest nontrivial cup power of this element—has been described for general m in the Ph.D. thesis [4] of the first author. Unfortunately, such a result gives no further information on $\text{Emb}(P^m)$ or, for that matter, on $\text{TC}^S(P^m)$ —this cup-power is just too low for $m \geq 7$. This

⁵This depends on the user's choice of generators x and y for D_8 right at the beginning of [14].

suggests the desirability of computing $h^*(B(P^m, 2))$ for other (richer) multiplicative cohomology theories. In such a generalized cohomology setting, (18) could play, together with the concept of *weight of a—generalized—cohomology class*, a more important role than in the current singular cohomology approach, cf. [8]. We intend to eventually come back to these ideas.

4 The cohomology ring $H^*(F(P^m, 2))$

A quick look at the Cartan-Leray spectral sequence for the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -action on $V_{m+1,2}$ in Definition 2.5 shows that $H^*(F(P^m, 2))$ has no odd torsion (cf. [12]). So, in this section we compute these integral cohomology groups via a thorough study of the 2-primary Bockstein spectral sequence (BSS) of $F(P^m, 2)$.

The first page of the BSS. The following description of the ring $H^*(F(P^m, 2); \mathbb{F}_2)$ was first brought to the authors' attention by Frederick Cohen. Recall the cohomology classes x_1 and y_1 introduced in the sentence following (2).

Lemma 4.1. *The map (3) induces an epimorphism $H^*(P^\infty \times P^\infty; \mathbb{F}_2) \rightarrow H^*(F(P^m, 2); \mathbb{F}_2)$ of rings whose kernel is the ideal generated by the three elements x_1^{m+1} , y_1^{m+1} , and $\sum x_1^i y_1^j$, where the summation runs over $i, j \geq 0$, $i + j = m$.*

Proof. The kernel of the morphism induced by the inclusion $P^m \times P^m \hookrightarrow P^\infty \times P^\infty$ is generated by x_1^{m+1} and y_1^{m+1} . The sum $\sum x_1^i y_1^j$ maps to the diagonal cohomology class in $P^m \times P^m$ in view of [22, Theorem 11.11]—which restricts to zero in $F(P^m, 2)$. So it suffices to check that the inclusion $F(P^m, 2) \hookrightarrow P^m \times P^m$ induces an epimorphism whose kernel is generated by the diagonal class. But (see [22, Section 11]) the map under consideration embeds into a long exact sequence

$$\cdots \rightarrow H^{*-m}(P^m; \mathbb{Z}_2) \rightarrow H^*(P^m \times P^m; \mathbb{Z}_2) \rightarrow H^*(F(P^m, 2); \mathbb{Z}_2) \rightarrow \cdots$$

(written here in terms of the Thom isomorphism for the normal bundle of the diagonal inclusion $P^m \hookrightarrow P^m \times P^m$). The desired conclusion follows from [22, Lemma 11.8] which shows that the map of degree m in this long exact sequence is given by multiplication by the diagonal class $\sum_{i+j=m} x_1^i y_1^j$ —a monomorphism in the current case. \square

First order Bocksteins. Lemma 4.1 implies that the monomials

$$x_1^i y_1^j \quad \text{with} \quad 0 \leq i \leq m, \quad 0 \leq j \leq m-1 \tag{24}$$

form an \mathbb{F}_2 -basis for the initial page of the BSS. Consider the filtration⁶ $0 = F^3 \subseteq F^2 \subseteq F^1 \subseteq F^0 = H^*(F(P^m, 2); \mathbb{F}_2)$ where F^k is generated by the basis elements in (24) with:

⁶This filtration was suggested by the referee.

- $j < m - 1$ if j is odd, for $k = 1$;
- even j , for $k = 2$.

(Note that $F^1 = F^0$ if m is odd.) The filtration is stable under the action of the first Bockstein Sq^1 , and we describe next the resulting “auxiliary” spectral sequence—converging to the second page of the BSS for $F(\mathbb{P}^m, 2)$. In what follows, the reader should keep in mind that the derivation Sq^1 is characterized by $\text{Sq}^1 a^k = k a^{k+1}$ for $a \in \{x_1, y_1\}$.

An \mathbb{F}_2 -basis for the Sq^1 -cohomology of F^2 is given by (the classes of) $1, y_1^2, \dots, y_1^{m-2+\delta}$ and, if m is odd, $x_1^m, x_1^m y_1^2, \dots, x_1^m y_1^{m-1}$. Here $m = 2t + \delta$ with $\delta \in \{0, 1\}$ —so that t is as in Theorem 2.1. Likewise, an \mathbb{F}_2 -basis for the Sq^1 -cohomology of F^1/F^2 is given by (the classes of) $y_1, y_1^3, \dots, y_1^{m-3+\delta}$ and, if m is odd, $x_1^m y_1, x_1^m y_1^3, \dots, x_1^m y_1^{m-2}$. Lastly, we have $F^0/F^1 = 0$ if m is odd, while for m even an \mathbb{F}_2 -basis for the Sq^1 -cohomology of F^0/F^1 is given by (the class of) $x_1^m y_1^{m-1}$. All these assertions are obvious, except for the last one which requires the following calculation in $H^*(F(\mathbb{P}^m, 2); \mathbb{F}_2)$: for even i with $0 \leq i \leq 2t - 2 = m - 2$,

$$\begin{aligned} \text{Sq}^1(x_1^i y_1^{2t-1}) &= x_1^i y_1^{2t} = x_1^i (x_1^{2t} + x_1^{2t-1} y_1 + \dots + x_1 y_1^{2t-1}) \\ &= x_1^{2t} y_1^i + x_1^{2t-1} y_1^{i+1} + \dots + x_1^{i+1} y_1^{2t-1} \\ &\equiv x_1^{i+1} y_1^{2t-1} \pmod{F^1}. \end{aligned}$$

The above considerations give the first page of the auxiliary spectral sequence. Note that, besides 1, $x_1^m y_1^{m-1}$ (for even m) and x_1^m (for odd m) represent permanent cycles in the auxiliary spectral sequence, for Lemma 4.1 gives in $H^*(F(\mathbb{P}^m, 2); \mathbb{F}_2)$

$$\begin{aligned} \text{Sq}^1(x_1^m y_1^{m-1}) &= x_1^m y_1^m = 0, \text{ for even } m; \\ \text{Sq}^1 x_1^m &= x_1^{m+1} = 0, \text{ for odd } m. \end{aligned}$$

All other classes in the auxiliary spectral sequence are wiped out by d_1 -differentials since, again in $H^*(F(\mathbb{P}^m, 2); \mathbb{F}_2)$,

$$\begin{aligned} \text{Sq}^1 y_1^j &= y_1^{j+1}, \quad \text{for odd } j, \ 1 \leq j \leq m - 3 + \delta; \\ \text{Sq}^1(x_1^m y_1^j) &= x_1^m y_1^{j+1}, \quad \text{for odd } j, \ 1 \leq j \leq m - 2 \quad (\text{relevant if } m \text{ is odd}). \end{aligned}$$

Thus, the auxiliary spectral sequence collapses from its second page which, as noted above, gives the second page of the BSS for $F(\mathbb{P}^m, 2)$. Further, the BSS collapses from its second page for dimensional reasons.

Immediate consequences. The BSS-analysis yields the following standard implications:

- (a) the torsion-free subgroups in $H^*(F(\mathbb{P}^m, 2))$ are as described in Theorem 2.2, with non-torsion positive-dimensional cohomology classes w_{2m-1} (for even m) and w_m (for odd m) having mod 2 reductions

$$\rho(w_{2m-1}) = x_1^m y_1^{m-1} \quad \text{and} \quad \rho(w_m) = x_1^m; \quad (25)$$

- (b) the torsion subgroups inject, via the mod 2 reduction map, into $H^*(F(P^m, 2); \mathbb{F}_2)$ with image that of the endomorphism

$$\text{Sq}^1: H^*(F(P^m, 2); \mathbb{F}_2) \rightarrow H^*(F(P^m, 2); \mathbb{F}_2). \quad (26)$$

This and Lemma 4.1 imply the first two items in Corollary 2.4, and lead (as indicated below) to the groups in Theorem 2.2. Yet, the finer multiplicative description (Theorems 2.1 and 2.3) requires a slightly more careful bookkeeping for the resulting classes in the image of (26). This is spelled out next in terms of the torsion elements $x_2, y_2, z_3 \in H^*(F(P^m, 2))$ defined in the sentence containing (3). We work directly with the basis elements in (24), keeping the notation $m = 2t + \delta$, $\delta \in \{0, 1\}$.

Additive counting. Consider the following partition of the basis elements in (24):

$$\begin{aligned} \mathcal{P}_0 &= \{ \text{basis elements in (24) for which } j \text{ is even and either } i \text{ is even or } i = m \}; \\ \mathcal{P}_1 &= \{ \text{basis elements in (24) for which } i \text{ and } j \text{ have distinct parity} \} - \mathcal{P}_0; \\ \mathcal{P}_2 &= \{ \text{basis elements in (24) for which both } i \text{ and } j \text{ are odd} \} - (\mathcal{P}_0 \cup \mathcal{P}_1). \end{aligned}$$

Elements in \mathcal{P}_0 can be ignored as they have trivial Sq^1 -image. A straightforward calculation shows that the set of Sq^1 -images of elements in \mathcal{P}_1 is formed by the basis elements

$$\rho(x_2^a y_2^b) = x_1^{2a} y_1^{2b} \quad \text{with } 0 \leq a \leq t, 0 \leq b \leq t - 1 + \delta, \text{ and } (a, b) \neq (0, 0) \quad (27)$$

and, when $\delta = 0$, by the (sum of basis) elements

$$\begin{aligned} \rho(x_2^a y_2^t) &= x_1^{2a} y_1^{2t} = x_1^{2a} (x_1^{2t} + x_1^{2t-1} y_1 + \cdots + x_1 y_1^{2t-1}) \\ &= x_1^{2t} y_1^{2a} + x_1^{2t-1} y_1^{2a+1} + \cdots + x_1^{2a+1} y_1^{2t-1} \end{aligned} \quad (28)$$

for $0 \leq a \leq t - 1$. Since the elements listed in (27) and (28) are linearly independent, this proves Theorem 2.3 and, by a simple counting, Theorem 2.2, both in even dimensions. Likewise, the set of Sq^1 -images of elements in \mathcal{P}_2 is formed by the linearly independent elements

$$\rho(x_2^a y_2^b z_3) = x_1^{2a+2} y_1^{2b+1} + x_1^{2a+1} y_1^{2b+2} \quad \text{with } 0 \leq a \leq t - 1 + \delta \text{ and } 0 \leq b \leq t - 2 + \delta. \quad (29)$$

[Note that, when $\delta = 0$, the previous assertion would seem to miss the Sq^1 -image of basis elements (24) of the form $x_1^{2a+1} y_1^{2t-1}$ with $0 \leq a \leq t - 1$. However, Lemma 4.1 gives

$$\begin{aligned} \text{Sq}^1(x_1^{2a+1} y_1^{2t-1}) &= x_1^{2a+2} y_1^{2t-1} + x_1^{2a+1} y_1^{2t} \\ &= x_1^{2a+2} y_1^{2t-1} + x_1^{2a+1} (x_1^{2t} + x_1^{2t-1} y_1 + \cdots + x_1 y_1^{2t-1}) \\ &= x_1^{2a+2} y_1^{2t-1} + x_1^{2t} y_1^{2a+1} + x_1^{2t-1} y_1^{2a+2} + \cdots + x_1^{2a+2} y_1^{2t-1} \\ &= (x_1^{2t} y_1^{2a+1} + x_1^{2t-1} y_1^{2a+2}) + \cdots + (x_1^{2a+4} y_1^{2t-3} + x_1^{2a+3} y_1^{2t-2}), \end{aligned}$$

which is a linear combination—trivial if $a = t - 1$ —of the elements in (29).] This proves Theorem 2.3 and, again by a simple counting, Theorem 2.2, now in odd dimensions.

Ring structure. It remains to prove Theorem 2.1. The two relations $w_{2m-1}^2 = 0$ (for even m) and $w_m^2 = 0$ (for odd m) are forced for dimensional reasons in view of Theorem 2.2. All other relations asserted in (4)–(6) involve exclusively torsion summands and, in view of the assertion containing (26), can be proved by reducing coefficients mod 2. Such a checking becomes a straightforward task (which is left to the reader) using Lemma 4.1, (2), and (25). The crux of the matter, then, lies in showing (in the next paragraphs) that these relations give a complete ring presentation for $H^*(F(P^m, 2))$.

For a positive integer m , consider the graded ring $\mathcal{R}_m = \mathbb{Z}[W, X, Y, Z]/I_m$ where W, X, Y, Z are formal variables of respective degrees $2m-1, 2, 2, 3$ for even m , and $m, 2, 2, 3$ for odd m , and where I_m is the ideal generated by polynomials $E = E(W, X, Y, Z)$ for which the corresponding element $e = E(w, x_2, y_2, z_3) \in H^*(F(P^m, 2))$ is one of the polynomial expressions on the left hand side of the relations listed in (1) and (4)–(6). Here we have written w for either w_{2m-1} or w_m , according to whether m is even or odd. For instance, for $m = 2t$, three of the generators of I_m are $Z^2 + XY(X + Y)$, W^2 , and $\sum X^i Y^j Z$, where the summation runs over $i, j \geq 0$ with $i + j = t - 1$. Thus, we have an epimorphism of rings $\Phi_m: \mathcal{R}_m \rightarrow H^*(F(P^m, 2))$, $\Phi_m(E) = e$. In order to show that this is a ring isomorphism (thus completing the proof of Theorem 2.1) it suffices to check that

the \mathbb{F}_2 -basis in Theorem 2.3 comes from generators for the torsion groups of \mathcal{R}_m . (30)

(Indeed, it is evident that Φ_m yields an isomorphism on the corresponding torsion-free subgroups, while the torsion subgroups of \mathcal{R}_m are \mathbb{F}_2 -vector spaces.)

We start with the \mathbb{Z} -basis of monomials $W^i X^j Y^k Z^\ell$, $i, j, k, \ell \geq 0$, for $\mathbb{Z}[W, X, Y, Z]$, and use each of the generators in I_m to rule out some of these basis elements—the (classes of the) remaining monomials will of course generate \mathcal{R}_m . In doing so, we can ignore all monomials W^i with $i \geq 0$, for we are focusing on torsion subgroups (so that the generators $2X, 2Y$, and $2Z$ of I_m are implicitly accounted for).

The generators W^2, X^{t+1}, Y^{t+1} , and $Z^2 + XY(X + Y)$ of I_m mean that our list of generating monomials reduces to

$$W^i X^j Y^k Z^\ell, \quad 0 \leq j, k \leq t, \quad 0 \leq i, \ell \leq 1 \quad (31)$$

where, as usual, $m = 2t + \delta$, $\delta \in \{0, 1\}$. Further, the generators in I_m which come from the relations in (6) and the last relation in (5), i.e. those involving w , imply that the restriction $0 \leq i \leq 1$ in (31) can in fact be strengthened to $i = 0$. Thus, in even dimensions we are left with the generating monomials $X^j Y^k$ with $0 \leq j, k \leq t$, $(j, k) \neq (0, 0)$, and, if $\delta = 0$, $(j, k) \neq (t, t)$ —in view of the generator $X^t Y^t$ of I_m for even m . This proves (30) in even

dimensions. On the other hand, in view of the generator $\sum X^j Y^k Z$ of I_m (the sum running over $j, k \geq 0$ with $j + k = t$), in odd dimensions we are left with the generating monomials $X^j Y^k Z$ with $0 \leq j \leq t$ and $0 \leq k \leq t - 1$, which completes the proof of (30) for odd m . Lastly, if m is even, the first of the two generators of I_m

$$\sum_{\substack{j+k=t-1 \\ j,k \geq 0}} X^j Y^k Z \quad \text{and} \quad \sum_{\substack{j+k=t \\ j,k \geq 0}} X^j Y^k Z \quad (32)$$

gives in \mathcal{R}_m the relations $X^t Z = -(X^{t-1} Y + \cdots + X Y^{t-1}) Z = Y^t Z$, so that the second generator in (32) can equivalently be replaced by $X^t Z$ (giving $Y^t Z \in I_m$ for free). Thus, this time in odd dimensions we are left with the generating monomials $X^j Y^k Z$ with $0 \leq j \leq t-1$ and $0 \leq k \leq t-2$, which completes the proof of (30) for even m .

5 The cohomology groups $H^*(B(\mathbb{P}^m, 2))$

As in the case of $H^*(F(\mathbb{P}^m, 2))$ in the previous section, the Cartan-Leray spectral sequence for the D_8 -action on $V_{m+1,2}$ in Definition 2.5 shows that $H^*(B(\mathbb{P}^m, 2))$ has no odd torsion—alternatively, use the corresponding property for $F(\mathbb{P}^m, 2)$, together with the transfer for the two-fold covering $F(\mathbb{P}^m, 2) \rightarrow B(\mathbb{P}^m, 2)$. Thus, in this section we make a thorough study of the 2-primary BSS of $B(\mathbb{P}^m, 2)$ in order to deduce the integral cohomology groups of $B(\mathbb{P}^m, 2)$.

The first page of the BSS. The following description of the ring $H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$ is proved in [13, Theorem 3.7]. Recall the cohomology classes u_1 , v_1 , and w_2 introduced in the sentence containing (8).

Lemma 5.1. *The map (10) induces an epimorphism $\beta^*: H^*(BD_8; \mathbb{F}_2) \rightarrow H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$ of rings with kernel the ideal generated by the two elements*

$$\sum_{i \geq 0} \binom{m-i}{i} v_1^{m-2i} w_2^i \quad \text{and} \quad \sum_{i \geq 0} \binom{m+1-i}{i} v_1^{m+1-2i} w_2^i. \quad (33)$$

Settling a basis for the mod 2 cohomology of $B(\mathbb{P}^m, 2)$ requires a bit more work than in the case of the $F(\mathbb{P}^m, 2)$ -analogue (24).

Lemma 5.2. *For $s = 0, 1, \dots, m$, the elements $R_{m+s} = \sum_{i \geq 0} \binom{m-s-i}{i} v_1^{m-s-2i} w_2^{s+i}$ vanish in $H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$.*

Proof. The first relation in (33) gives $R_m = 0$. The relation $R_{m+1} = 0$ follows by adding the second element in (33) to the v_1 -multiple of the first element in (33)—and pulling back under β . The rest of the relations then follow inductively by noticing that $R_{m+s} = v_1 R_{m+s-1} + w_2 R_{m+s-2}$. \square

Corollary 5.3. *An \mathbb{F}_2 -basis for $H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$ is given by the monomials*

$$u_1^\varepsilon v_1^r w_2^s \quad \text{with } \varepsilon \leq 1 \quad \text{and } r + s < m. \quad (34)$$

Proof. In view of (8) and the relations $R_{m+s} = v_1^{m-s} w_2^s + \cdots = 0$ in Lemma 5.2, the indicated elements are additive generators. Linear independence follows from the next result, since the number of monomials in (34) matches the (graded-wise) \mathbb{F}_2 -dimension of $H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$. \square

Sublemma 5.4. *For any m ,*

$$H^i(B(\mathbb{P}^m, 2); \mathbb{F}_2) = \begin{cases} \langle i+1 \rangle, & 0 \leq i \leq m-1; \\ \langle 2m-i \rangle, & m \leq i \leq 2m-1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The assertion for $i \geq 2m$ follows from the fact that $B(\mathbb{P}^m, 2)$ has the homotopy type of the closed $(2m-1)$ -dimensional manifold $V_{m+1,2}/D_8$ (see Definition 2.5). Poincaré duality then gives the assertion for $m \leq i \leq 2m-1$ as a consequence of that for $0 \leq i \leq m-1$. Lastly, the assertion for $0 \leq i \leq m-1$ follows from Lemma 5.1 and the fact that $H^i(BD_8; \mathbb{F}_2) = \langle i+1 \rangle$; indeed, the presentation (8) implies that an \mathbb{F}_2 -basis for $H^*(BD_8; \mathbb{F}_2)$ is given by all monomials $u_1^\varepsilon v_1^r w_2^s$ with $\varepsilon \leq 1$ (this basis will be in force throughout the next considerations). \square

The auxiliary spectral sequence. The Sq^1 -action on $H^*(BD_8; \mathbb{F}_2)$ is implicit in [13, Proposition 3.5]: $\text{Sq}^1(w_2) = v_1 w_2$, and $\text{Sq}^1(\xi_1) = \xi_1^2$ for $\xi_1 \in \{u_1, v_1\}$. The Cartan formula then yields

$$\text{Sq}^1(u_1^\varepsilon v_1^r w_2^s) = (\varepsilon + r + s) u_1^\varepsilon v_1^{r+1} w_2^s, \quad (35)$$

which also holds in $H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$ by naturality. Consider the filtration $0 = B^m \subseteq B^{m-1} \subseteq \cdots \subseteq B^1 \subseteq B^0 = H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$ where B^k is generated by the basis elements in (34) with $s \geq k$. Each B^k is stable under the action of Sq^1 in view of Lemma 5.2 and (34). We describe next the resulting “auxiliary” spectral sequence—converging to the second page of the BSS for $B(\mathbb{P}^m, 2)$.

For $k = 0, \dots, m-1$, a basis for B^k/B^{k+1} is given by the monomials (34) with $s = k$ and, in these terms, the filtered Sq^1 -action takes the form

$$\text{Sq}^1(u_1^\varepsilon v_1^r w_2^k) = \begin{cases} (\varepsilon + r + k) u_1^\varepsilon v_1^{r+1} w_2^k, & r + k < m-1; \\ 0, & r + k = m-1, \end{cases}$$

in view of (35) and Lemma 5.2. Then, an \mathbb{F}_2 -basis for the cycles in the 0-th page of the auxiliary spectral sequence is given by the monomials in (34) for which either $r + s = m-1$

or $\varepsilon + r + s$ is even. Likewise, an \mathbb{F}_2 -basis for the corresponding boundaries is given by the monomials in (34) for which

$$r > 0 \quad \text{and} \quad \varepsilon + r + s \equiv 0 \pmod{2}. \quad (36)$$

Thus, an \mathbb{F}_2 -basis for the first page of the auxiliary spectral sequence is given by the monomials in (34) for which one of the following two conditions holds:

- (a) $r + s = m - 1$, and either $r = 0$ or $\varepsilon + r + s$ is odd.
- (b) $r = 0$, $s < m - 1$, and $\varepsilon + s$ is even.

The explicit elements of type (b) are $u_1^\varepsilon w_2^s$ for $0 \leq s \leq m - 2$ and $\varepsilon + s \equiv 0 \pmod{2}$, all of which are permanent cycles in the auxiliary spectral sequence in view of (35). On the other hand, the explicit elements of type (a) are $u_1 w_2^{m-1}$, w_2^{m-1} and

$$u_1^\varepsilon v_1^{m-s-1} w_2^s \quad \text{for } 0 \leq s \leq m - 2 \quad \text{and} \quad \varepsilon + m \equiv 0 \pmod{2}. \quad (37)$$

We show next that most of these $m + 1$ elements are wiped out by d_1 -differentials in the auxiliary spectral sequence, whereas the few d_1 -cycles which are not d_1 -boundaries are in fact permanent cycles.

Case m even: (Note that $\varepsilon = 0$ in (37).) The differentials

$$d_1(v_1^{m-2i-1} w_2^{2i}) = v_1^{m-2i-2} w_2^{2i+1}, \quad 0 \leq i \leq \frac{m}{2} - 1, \quad (38)$$

hold since (35) and Lemma 5.2 give $\text{Sq}^1(v_1^{m-2i-1} w_2^{2i}) = v_1^{m-2i} w_2^{2i} \equiv v_1^{m-2i-2} w_2^{2i+1} \pmod{B^{2i+2}}$. On the other hand, the only element of type (a) not considered in the above d_1 -differentials, namely $u_1 w_2^{m-1}$, is in fact a permanent cycle in view of (35).

Case m odd: (Note that $\varepsilon = 1$ in (37).) The differentials

$$d_1(u_1 v_1^{m-2i} w_2^{2i-1}) = u_1 v_1^{m-2i-1} w_2^{2i}, \quad 1 \leq i \leq \frac{m-1}{2}, \quad (39)$$

hold since (35) and Lemma 5.2 give $\text{Sq}^1(u_1 v_1^{m-2i} w_2^{2i-1}) = u_1 v_1^{m-2i+1} w_2^{2i-1} \equiv u_1 v_1^{m-2i-1} w_2^{2i} \pmod{B^{2i+1}}$. On the other hand, the only two elements of type (a) not considered in the above d_1 -differentials, namely $u_1 v_1^{m-1}$ and w_2^{m-1} , are in fact permanent cycles. Indeed, the assertion is obvious from (35) in the case of w_2^{m-1} . For $u_1 v_1^{m-1}$ use (35) and Lemma 5.2 to get

$$\text{Sq}^1(u_1 v_1^{m-1}) = u_1 v_1^m = \sum_{i \geq 1} \binom{m-i}{i} u_1 v_1^{m-2i} w_2^i,$$

and note that $\binom{m-i}{i}$ is even if i is odd, whereas $u_1 v_1^{m-2i} w_2^i = \text{Sq}^1(u_1 v_1^{m-2i-1} w_2^i)$ if i is even. So, the element

$$u_1 v_1^{m-1} + \sum_{i \geq 1} \binom{m-2i}{2i} u_1 v_1^{m-4i-1} w_2^{2i} \quad (40)$$

is a permanent cycle in the auxiliary spectral sequence representing the same class as $u_1 v_1^{m-1}$.

Second order Bocksteins. We have proved that an \mathbb{F}_2 -basis of the second page of the BSS of $B(P^m, 2)$ is represented by the monomials

$$u_1^\varepsilon w_2^s, \quad 0 \leq s \leq m-1, \quad \varepsilon + s \equiv 0 \pmod{2}, \quad (41)$$

together with an extra basis element represented by (40) if m is odd. Next we analyze the second Bockstein differentials in $B(P^m, 2)$ and, for this purpose, we begin by taking a look at the BSS of BD_8 . Observe from (35) that an \mathbb{F}_2 -basis for the second page of the BSS for BD_8 is represented by the monomials $u_1^\varepsilon w_2^s$ with $\varepsilon + s \equiv 0 \pmod{2}$. Furthermore, the family of second Bockstein differentials

$$\beta_2(u_1 w_2^{2\ell-1}) = w_2^{2\ell} \quad \text{for } \ell \geq 1 \quad (42)$$

follows from the fact that the only 4-torsion classes in $H^*(BD_8)$ come from the powers d_4^ℓ , which are concentrated in positive dimensions congruent to zero modulo 4. In particular, the third page of the BSS for BD_8 is concentrated in degree 0, forcing its collapse from this page on. Now, the β_2 -differentials in (42) pull back under the map in (10) to yield a corresponding family of second Bockstein differentials in $B(P^m, 2)$; this wipes all of the monomials in (41), except for those with $s = 0$ and, for even m , $s = m-1$. On the other hand, if $m \equiv 1 \pmod{4}$, the element in (40) has trivial β_2 -differential because none of the elements in (41) lies in a dimension congruent to 2 mod 4. In any case, only two classes survive to the third page of the BSS of $B(P^m, 2)$: $1 = u_1^0 w_2^0$ and a class represented by

- (i) $u_1 w_2^{m-1}$, if m is even;
- (ii) either (40) or the sum of (40) with $u_1 w_2^t$, if $m = 2t + 1$.

The actual representative in (ii) depends on whether the second Bockstein of (40) is trivial or not. [As noted above, (40) alone gives the right representative if t is even; however, the final considerations in Section 6 imply that the extra summand $u_1 w_2^t$ is actually needed for odd t .] The BSS of $B(P^m, 2)$ collapses from this point on for dimensional reasons.

Immediate consequences. The above BSS-analysis has a number of standard implications. First, we see that the torsion-free subgroups in $H^*(B(P^m, 2))$ are as described in Theorem 2.7, with a torsion-free positive-dimensional cohomology generator, e_{2m-1} for even

m , and e_m for odd m . Their mod 2 reductions are described (partially⁷, for $m \equiv 3 \pmod{4}$) in (i) and (ii) above. Second, multiplication by 4 kills the torsion subgroups in the integral cohomology of $B(\mathbb{P}^m, 2)$. Next, not only does the map (10) give a surjection on the first page of the corresponding BSS's (Lemma 5.1), but on positive degrees of the second page level, it maps onto non-permanent cycles. Together with the collapse of both spectral sequences from their third pages on, this yields the first two items in Corollary 2.9. The last of the immediate consequences of our BSS-analysis for $B(\mathbb{P}^m, 2)$ is that we have a good hold on the number of direct summands \mathbb{Z}_2 and \mathbb{Z}_4 in the integral cohomology of $B(\mathbb{P}^m, 2)$. Indeed, these are given by the \mathbb{F}_2 -dimension of the images of the first and second Bockstein differentials, respectively. The explicit counting of dimensions (which yields the proof of Theorem 2.7) is done in the next paragraphs.

Additive counting. In view of (41) and (42), an \mathbb{F}_2 -basis for the β_2 -image is given by the monomials w_2^{2i} for $1 \leq i \leq m - 2 + \delta$ where $m = 2t + \delta$, $\delta \in \{0, 1\}$ (note that the β_2 -indeterminacy inherent in (ii) above does not play a role here). Therefore, there is a single \mathbb{Z}_4 -summand only in each positive dimension n satisfying $n < 2m - 1$ and $n \equiv 0 \pmod{4}$.

Counting the \mathbb{F}_2 -dimension of the Sq^1 -image gets (combinatorially) more involved, but the task is simplified by working in terms of the auxiliary spectral sequence. Level-0 (i.e. filtered) Sq^1 -boundaries have \mathbb{F}_2 -basis given by the monomials in (34) satisfying (36); level-1 Sq^1 -boundaries (i.e. d_1 -differentials in the auxiliary spectral sequence) have the \mathbb{F}_2 -basis indicated on the right hand side of the equations in (38) and (39). Since there are no higher-level Sq^1 -boundaries (i.e. higher differentials), we find that, up to elements of higher auxiliary filtration (a proviso which is irrelevant for the purpose of counting \mathbb{F}_2 -dimensions), an \mathbb{F}_2 -basis for the Sq^1 -boundaries consists of the monomials $u_1^\varepsilon v_1^r w_2^s$ satisfying one of the following two (disjoint) sets of conditions:

$$\varepsilon \leq 1, \quad r + s < m, \quad r > 0, \quad \text{and} \quad \varepsilon + r + s \equiv 0 \pmod{2}; \quad (43)$$

$$\varepsilon = \delta, \quad r = m - 2i - 2 + \delta, \quad \text{and} \quad s = 2i + 1 - \delta, \quad \text{for} \quad \delta \leq i \leq t - 1 + \delta. \quad (44)$$

Theorem 2.7 now follows from a dimension-wise count of the above basis elements. The required checking is straightforward, but the legwork comes from the large number of cases to consider. For the reader's benefit, we illustrate the type of counting needed by working out a representative case, namely the one corresponding to the eighth line in the description of $H^*(B(\mathbb{P}^{2t+1}, 2))$ in Theorem 2.7: We want to count the number of basis elements $u_1^\varepsilon v_1^r w_2^s$ satisfying (43) or (44), as well as

$$m = 2t + 1 < \dim(u_1^\varepsilon v_1^r w_2^s) = 4a + 1 \leq 4t + 1. \quad (45)$$

Note that the equality in (45) and the last condition in (43) force s to be odd, so the equality in (45) becomes $1 \equiv 2 + r + \varepsilon \pmod{4}$. This happens only for $r \equiv 2 \pmod{4}$ (with

⁷The indeterminacy will be removed in Section 6.

$\varepsilon = 1$) or $r \equiv 3 \pmod{4}$ (with $\varepsilon = 0$). Thus, the actual possibilities for the pair (ε, r) are $(1, 4i - 2)$ and $(0, 4i - 1)$ —both with $s = 2a - 2i + 1$ in view of the equality in (45)—for $1 \leq i \leq t - a$, where the latter inequality comes from the second condition in (43). Therefore, there are $2(t - a)$ basis elements in dimension $4a + 1$ accounted for by (43). The extra basis element reported by the group $\langle 2t + 1 - 2a \rangle$ in Theorem 2.7 comes from (44), where the dimensional hypothesis in (45) becomes $i = 2a - t$ (this is in the range indicated in (44), in view of (45)).

6 The ring structure of $H^*(B(P^m, 2))$

We now prove Theorems 2.6 and 2.8. Unlike the case of $F(P^m, 2)$, where the proof of Theorem 2.1 uses the auxiliary algebraic model \mathcal{R}_m , proofs in this section depend on a very explicit handling of relations in the torsion subgroups of the integral cohomology ring of $B(P^m, 2)$. In particular, the method in the final part of this section (proof of Theorem 2.8) is similar to the deduction of the relations R_{m+s} in Lemma 5.2 and their use in Corollary 5.3 for easily obtaining an additive basis for $H^*(B(P^m, 2); \mathbb{F}_2)$.

The relations: simplifying considerations. The equations in (12) and (13) corresponding to $d_4^{t+\delta} = 0$ follow from dimensional considerations. This is also the case for the family of equations in (12) involving e_{2m-1} . Further, the first three equations in (13) follow respectively from the three relations in (11) because the maps in (10) are compatible under the equatorial inclusion $B(P^{2t+1}, 2) \hookrightarrow B(P^{2t+2}, 2)$. We now focus on

$$\text{the three equations in (11) and the first two equations in (12).} \quad (46)$$

A straightforward calculation (left to the reader) using (7)–(9), (33), and Lemma 5.2 shows that the equations in (46) hold after applying the mod 2 reduction morphism $\rho: H^*(B(P^m, 2)) \rightarrow H^*(B(P^m, 2), \mathbb{F}_2)$. The latter map is monic on torsion elements of dimension not divisible by 4 (where there are no copies of \mathbb{Z}_4), so that the equations in (46) lying in dimensions not divisible by 4 already hold in $H^*(B(P^m, 2))$. As for the equations in (46) that lie in dimensions divisible by 4, note that:

- the equatorial inclusion $B(P^{2t}, 2) \hookrightarrow B(P^{2t+1}, 2)$ induces a cohomology epimorphism in even dimensions (Corollary 2.9.1), and
- the groups $H^*(B(P^{2t}, 2))$ and $H^*(B(P^{2t+1}, 2))$ are isomorphic in even dimensions not greater than $4t - 1$ (Theorem 2.7).

So, the only equations in (46) actually requiring direct verification are

$$a_2\sigma_{2t} = 0 \quad \text{and} \quad b_2\sigma_{2t} + \iota_{2t+2} = 0 \quad \text{for } t \text{ odd, } t \geq 3; \quad (47)$$

$$b_2d_4\sigma_{2t-2} + \iota_{2t+4} = 0 \quad \text{for } t \text{ even, } t \geq 4, \quad (48)$$

all of these with $\delta = 0$ (i.e. as elements of $H^*(B(P^{2t}, 2))$), as well as

$$a_2\sigma_{2t} = 0 \quad \text{and} \quad b_2\sigma_{2t} + \iota_{2t+2} = 0 \quad \text{for } t = 1; \quad (49)$$

$$b_2d_4\sigma_{2t-2} + \iota_{2t+4} = 0 \quad \text{for } t = 2, \quad (50)$$

all of these with $\delta = 1$ (i.e. as elements of $H^*(B(P^{2t+1}, 2))$).

The relations: strategy of proof. Equations (47)–(50) can be approached⁸ through the commutative diagram

$$\begin{array}{ccccc} H^{*-1}(BD_8) & \xrightarrow{\rho} & H^{*-1}(BD_8; \mathbb{F}_2) & \xrightarrow{\partial} & H^*(BD_8) \\ \beta^* \downarrow & & \beta^* \downarrow & & \beta^* \downarrow \\ H^{*-1}(B(P^m, 2)) & \xrightarrow{\rho} & H^{*-1}(B(P^m, 2); \mathbb{F}_2) & \xrightarrow{\partial} & H^*(B(P^m, 2)) \end{array} \quad (51)$$

where the rows are portions of the long exact sequences giving the corresponding BSS's. Namely, exactness implies that the triviality of an element $\zeta \in H^*(B(P^m, 2))$ with $2\zeta = 0$ —i.e. in the image of the boundary operator of the bottom row—is established by showing that ζ lies in the image of the composite lower row. Such a task can be carried out in terms of the composite top row: it suffices to find elements $\xi \in H^{*-1}(BD_8)$ and $\eta \in H^{*-1}(BD_8; \mathbb{F}_2)$ with

$$\rho(\xi) \equiv \eta \pmod{\text{Ker}(\beta^*)} \quad \text{and} \quad \beta^*(\partial(\eta)) = \zeta. \quad (52)$$

The point is that the top row in (51) is fully accessible in view of (7)–(9) and the fact (Lemma 6.1 below) that the connecting morphism

$$\partial: H^{*-1}(BD_8; \mathbb{F}_2) \rightarrow H^*(BD_8) \quad (53)$$

is well understood in terms of the Wall-Hamada resolution for the trivial D_8 -module \mathbb{Z} . The relevant information can be found in [14] (see particularly Proposition 4.3, Equation (5.1), and the proofs of Theorems 5.2 and 5.5), where a fairly complete description of the multiplicative properties of the cohomology of D_8 is presented in great detail. The explicit result we need is:

Lemma 6.1 ([14]). *The connecting map in (53) is characterized by*

$$\partial(u_1^\varepsilon v_1^{2i_1+\varepsilon_1} w_2^{2i_2+\varepsilon_2}) = \begin{cases} \varepsilon a_2^{i_1} b_2 d_4^{i_2}, & \varepsilon_1 = \varepsilon_2 = 0; \\ \varepsilon a_2^{i_1} b_2 c_3 d_4^{i_2}, & \varepsilon_1 = \varepsilon_2 = 1; \\ (1 + \varepsilon) a_2^{i_1+1} d_4^{i_2}, & \varepsilon_1 = 1 \text{ and } \varepsilon_2 = 0; \\ (1 + \varepsilon) a_2^{i_1} c_3^{1-\varepsilon} d_4^{i_2+\varepsilon}, & \varepsilon_1 = 0 \text{ and } \varepsilon_2 = 1, \end{cases}$$

⁸The idea can be used to verify most of the relations claimed in Theorem 2.6 (e.g. all of the equations in (46)—except for the second equation in (49), see below), but the legwork is conveniently reduced by the above ‘simplifying considerations’.

for integers $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{0, 1\}$ and $i_1, i_2 \geq 0$.

Note that $\partial(u_1 v_1^{2i_1} w_2^{2i_2+1}) = 0$ for $i_1 > 0$, but $\partial(u_1 w_2^{2i_2+1}) = 2d_4^{i_2+1}$. This behavior leads to the summands “ ι_{2t+2} ” and “ ι_{2t+4} ” in (47)–(50).

The relations: main computation instructions. Elements satisfying (52) can be chosen as follows:

- For $\zeta = a_2 \sigma_{2t}$ with $t = 2\ell + 1$, $\ell \geq 1$, and $\delta = 0$, take

$$\eta = \sum_{j=0}^{\ell} \binom{2t-2j}{2j} v_1^{2t+1-4j} w_2^{2j} \quad \text{and} \quad \xi = \sum_{j=0}^{\ell} \binom{2t-1-2j}{2j+1} a_2^{t-1-2j} c_3 d_4^j.$$

The term in $\text{Ker}(\beta^*)$ needed in (52) is the v_1 -multiple of the first sum in (33).

- For $\zeta = b_2 \sigma_{2t} + \iota_{2t+2}$ with $t = 2\ell + 1$, $\ell \geq 1$, and $\delta = 0$, take

$$\eta = u_1 w_2^t + \sum_{j=0}^{\ell} \binom{2t-2j}{2j} u_1 v_1^{2t-4j} w_2^{2j} \quad \text{and} \quad \xi = \sum_{j=0}^{\ell-1} \binom{2t-1-2j}{2j+1} a_2^{t-2-2j} b_2 c_3 d_4^j.$$

The term in $\text{Ker}(\beta^*)$ needed in (52) is the u_1 -multiple of the first sum in (33).

- For $\zeta = b_2 d_4 \sigma_{2t-2} + \iota_{2t+4}$ with $t = 2\ell$, $\ell \geq 2$, and $\delta = 0$, take

$$\eta = u_1 w_2^{t+1} + \sum_{j=0}^{\ell-1} \binom{2t-2-2j}{2j} u_1 v_1^{2t-2-4j} w_2^{2+2j}$$

and

$$\xi = \sum_{j=0}^{\ell-2} \binom{2t-3-2j}{2j+1} a_2^{t-3-2j} b_2 c_3 d_4^{j+1}.$$

The term in $\text{Ker}(\beta^*)$ needed in (52) is the u_1 -multiple of R_{m+2} in Lemma 5.2.

- For $\zeta = a_2 \sigma_{2t}$ with $t = \delta = 1$, take $\eta = v_1^3$ and $\xi = 0$. The term in $\text{Ker}(\beta^*)$ needed in (52) is the first sum in (33).
- For $\zeta = b_2 d_4 \sigma_{2t-2} + \iota_{2t+4}$ with $t = 2$ and $\delta = 1$, take $\eta = u_1 v_1^2 w_2^2 + u_1 w_2^3$ and $\xi = a_2 b_2 c_3$. The term in $\text{Ker}(\beta^*)$ needed in (52) is the u_1 -multiple of R_{m+1} in Lemma 5.2.

The second equation in (49)—the only equation among those involving only torsion elements, and that we have not yet indicated how to check—is exceptional: the method

fails to verify it because the left-most vertical map in (51) is not surjective (we deal below with this case). Indeed, although the ∂ -image of the element

$$u_1 v_1^2 + u_1 w_2 \in H^3(B(\mathbb{P}^3, 2); \mathbb{F}_2) \quad (54)$$

is $b_2 \sigma_2 + \iota_4 \in H^*(B(\mathbb{P}^m, 2))$ —the element asserted to be trivial—, any ρ -preimage of (54) involves the torsion-free class e_3 , an element not in the image of the right-most vertical map in (51). To clarify this, note that (23), Lemma 3.4, Theorem 9.1 in [10], and the fact (coming from Theorem 2.7) that $H^4(B(\mathbb{P}^3, 2))$ is a cyclic group of order 4 (necessarily generated by d_4) imply the relation $b_2^2 = 2d_4$ in this group. This is the second equation in (49) in view of (7), thus completing the verification of (11)–(13). But more importantly, the new information can be used to shed light on the above viewpoint. Namely, exactness of the bottom row in (51) implies that the element in (54) does lie in the image of the mod 2 reduction map $\rho: H^3(B(\mathbb{P}^3, 2)) \rightarrow H^3(B(\mathbb{P}^3, 2); \mathbb{F}_2)$. But $H^3(B(\mathbb{P}^3, 2)) = \mathbb{Z} \oplus \mathbb{Z}_2$, where c_3 —the generator of the torsion subgroup—has $\rho(c_3) = v_1 w_2$ in view of (9). Since an \mathbb{F}_2 -basis for $H^3(B(\mathbb{P}^3, 2); \mathbb{F}_2)$ is given by the three elements $u_1 v_1^2$, $v_1 w_2$, and $u_1 w_2$ (Corollary 5.3), the torsion-free class e_3 in Theorem 2.6(b) can actually be chosen to have (54) as its mod 2 reduction. In particular, since $H^5(B(\mathbb{P}^3, 2)) = \mathbb{Z}_2$ (so that the mod 2 reduction map $\rho: H^*(B(\mathbb{P}^3, 2)) \rightarrow H^*(B(\mathbb{P}^3, 2); \mathbb{F}_2)$ is injective in dimension 5) and $H^k(B(\mathbb{P}^3, 2)) = 0$ for $k \geq 6$, the relations in (14) are easily proved for $m = 3$ by checking them after applying the mod 2 reduction map.

The same idea will be used below to verify the equations in (14) for general (odd) m —the only relations we have not yet indicated how to verify. As for $m = 3$, the explicit calculations require making a choice for the integral classes in Theorem 2.6, which in turn depends on a description of minimal additive generators for $H^*(B(\mathbb{P}^m, 2))$ —a task whose solution we explain next.

Minimal additive generators and ring presentation. For $0 \leq s \leq r$ let $R_{r,s}$ stand for the element $\sum_{i \geq 0} \binom{r-s-i}{i} a_2^{r-s-2i} d_4^{s+i} \in H^{2r+2s}(BD_8)$ as well as its image under the map $\beta: B(\mathbb{P}^m, 2) \rightarrow BD_8$ in (10). There are identities

$$R_{r,0} = \sigma_{2r}, \quad R_{r,1} = \sigma_{2r+2} - a_2 \sigma_{2r}, \quad \text{and} \quad R_{r,s+2} = d_4 R_{r,s} - a_2 R_{r,s+1} \quad (55)$$

where the first one holds by definition, and the last two are based on the binomial identity $\binom{a}{b} = \binom{a+1}{b+1} - \binom{a}{b+1}$. The next result uses the elements ι_{2r} in Theorem 2.6.

Lemma 6.2. *Let $m = 2t + \delta$, $\delta \in \{0, 1\}$. The following elements vanish in $H^*(B(\mathbb{P}^m, 2))$:*

1. $a_2 R_{t,s}$ and $b_2 R_{t,s} + \iota_{2t+2s+2}$, for $0 \leq s \leq t$;
2. $c_3 R_{t-1+\delta,s}$ for $0 \leq s \leq t - 1 + \delta$.

Proof. This is an easy exercise using the relations (11)–(13) and (55)—in the case of $a_2 R_{t,s}$, note that the c_3 -multiple of the first equation in (12) becomes $a_2 R_{t,1} = 0$ in view of the last equation in (7). \square

Let $H^*(m)$ be the subring of $H^*(B(\mathbb{P}^m, 2))$ generated by the classes a_2, b_2, c_4, d_4 . Thus, besides the unit $1 \in H^0(m)$, $H^*(m)$ consists of all the torsion elements in $H^*(B(\mathbb{P}^m, 2))$. Alternatively, $H^*(m)$ is the image of the morphism induced by the map in (10).

Proof of Theorem 2.8. The monomials $a_2^i b_2^\varepsilon c_3^{\varepsilon'} d_4^j$ with $i, j \geq 0$ and $\varepsilon, \varepsilon' \in \{0, 1\}$ are additive generators for $H^*(BD_8)$ in view of (7). Corollary 2.9, Lemma 6.2, and the relation $d_4^{t+\delta} = 0$ in (12) and (13) then imply that the elements in (15) are additive generators for $H^*(m)$ in positive dimensions. Thus, the proof reduces to checking that the elements in (15) give the right size for the groups reported in Theorem 2.7. Such a task requires a dimension-wise count analogous to that of the basis elements $u_1^\varepsilon v_1^r w_2^s$ satisfying (43) or (44). [The current counting gives more precise information than the one noted at the end of Section 5 since the latter one is performed on elements capturing integral cohomological information only up to higher auxiliary filtration.] The counting needed now is rather simple, and we omit the straightforward details. Yet, for the reader's convenience, Example 6.3 below deals with a couple of representative cases, namely the ones corresponding to the $(6 + \delta)$ -th line in the description of $H^*(B(\mathbb{P}^{2t+\delta}, 2))$ in Theorem 2.7. \square

Example 6.3. In dimensions 4ℓ with $2t + 1 < 4\ell \leq 4t + 1$, the monomials in (15) take either one of the forms $a_2^{2i} d_4^{\ell-i}$ and $a_2^{2i-1} b_2 d_4^{\ell-i}$ for $0 \leq i \leq t - \ell$. For $i > 0$, these give $2(t - \ell)$ elements of order 2, whereas the case $i = 0$ —giving the element d^ℓ —accounts for a \mathbb{Z}_4 -group.

The above argument also shows that $H^*(m)$ is presented as a ring as indicated in Theorem 2.8, except that one has to remove the relations involving the torsion-free positive-dimensional classes e_{2m-1} (for even m) and e_m (for odd m).

Proof of Theorem 2.6—sketch of conclusion. It remains to verify the relations in (14), that is, the instructions for multiplying with the torsion-free class $e_m \in H^m(B(\mathbb{P}^m, 2))$ in Theorem 2.6(b). As a first step we choose explicit generators for all positive-dimensional \mathbb{Z} -groups.

As $H^{4t-1}(B(\mathbb{P}^{2t}, 2)) = \mathbb{Z}$, there is no real choice to make (except for sign) for $m = 2t$: Corollary 5.3 forces

$$\rho(e_{2m-1}) = u_1 w_2^{m-1}, \quad (56)$$

which is the only nonzero element in $H^{4t-1}(B(\mathbb{P}^{2t}, 2); \mathbb{F}_2) = \mathbb{Z}_2$ —(56) has also been noted at the end of the paragraph ‘Second order Bocksteins’ in Section 5.

The situation for m odd is not as direct, but can still be analyzed using (51) with $* = m + 1$. Namely, Theorem 2.8 and Corollary 5.3 give explicit minimal generators

(actual \mathbb{F}_2 -basis if no \mathbb{Z}_4 -summands are involved) for the torsion subgroups of the groups in the lower row of (51), whereas (9), Corollary 2.9.1, and Lemma 6.1 can be used to describe the morphisms between these groups. The morphisms behave transparently on bases, sending basis elements to zero or to other basis elements, except for the basis element $u_1 v_1^{m-1} \in H^m(B(\mathbb{P}^m, 2); \mathbb{F}_2)$. Indeed, the second relation in (11) is needed to express $\partial(u_1 v_1^{m-1})$ as a linear combination of minimal generators in $H^{m+1}(B(\mathbb{P}^m, 2))$. This yields detailed \mathbb{F}_2 -bases for the kernel of ∂ and for the image under ρ of the torsion subgroup of $H^m(B(\mathbb{P}^m, 2))$ and, as a result, an element is singled out in the former kernel-group which is not in the latter image-group. Then, just as in the case $m = 3$ discussed right after (54), exactness of the lower row in (51) implies that the singled-out element must be the mod 2 reduction of a torsion-free class e_m . The reader is encouraged to fill in the easy details verifying the above discussion, and we content ourselves with reporting the net outcome: For $m = 2t + 1$, the class e_m in Theorem 2.6(b) can be chosen to have

$$\rho(e_m) = \sum_{i \geq 0} \binom{t-i}{i} u_1 v_1^{2t-4i} w_2^{2i} + t u_1 w_2^t. \quad (57)$$

Note this is in agreement—and refines—the considerations at the end of the paragraph ‘Second order Bocksteins’ in Section 5.

The remainder of the proof is standard: The first relation in (14), as well as the last two for $m \leq 3$, hold for dimensional reasons (note that the sum in (14) is empty if $m \leq 3$, whereas the relation $0 = b_2 R_{1,1} + \iota_6$ in Lemma 6.2.1 gives the triviality of $b_2 d_4$, the right-hand-side term in the third relation in (14) for $m = 3$). For the rest of the relations one first shows, by straightforward calculation (see Example 6.4 below), that they hold after evaluating under the mod 2 reduction map $\rho: H^*(B(\mathbb{P}^m, 2)) \rightarrow H^*(B(\mathbb{P}^m, 2); \mathbb{F}_2)$. As this map is monic on torsion elements of dimension not divisible by four, the asserted relations in $H^*(B(\mathbb{P}^m, 2))$ hold for free, except for the third relation in (14) if $m \equiv 1 \pmod{4}$. Indeed, if $\ell \geq 1$, the kernel of $\rho: H^{4\ell+4}(B(\mathbb{P}^{4\ell+1}, 2)) \rightarrow H^{4\ell+4}(B(\mathbb{P}^{4\ell+1}, 2); \mathbb{F}_2)$ is a copy of \mathbb{Z}_2 generated by $2d_4^{\ell+1}$, so that all we have here is $c_3 e_{4\ell+1} = \eta d_4^{\ell+1}$ for $\eta \in \{0, 2\}$. To solve the indeterminacy (for $m \neq 5$), compute

$$\eta d_4^{\ell+2} = c_3(d_4 e_{4\ell+1}) = c_3 \sum_{i=1}^{\ell} \binom{2\ell-i}{i-1} a_2^{2\ell-2i} b_2 c_3 d_4^i = \sum_{i=1}^{\ell} \binom{2\ell-i}{i-1} a_2^{2\ell+1-2i} b_2 d_4^{i+1}$$

and note that the last sum is the d_4 -multiple of the left-hand-side term of the relation $b_2 R_{2\ell,1} = \iota_{4\ell+4}$ in Lemma 6.2.1. This yields $\eta d_4^{\ell+2} = 2d_4^{\ell+2}$ or, equivalently (as $d_4^{\ell+2}$ is of order 4 if $\ell \geq 2$), $\eta = 2$. \square

Example 6.4. We verify in detail the last relation in (14). Recall $m = 2t + 1$ and $t = 2\ell + \kappa$ with $\kappa \in \{0, 1\}$. Use (9), (57), and Lemma 5.2 (with $s = 1$) to get

$$\begin{aligned}
\rho(d_4 e_m) &= \sum_{i=0}^{\ell} \binom{t-i}{i} u_1 v_1^{2t-4i} w_2^{2i+2} + t u_1 w_2^{t+2} \\
&= u_1 v_1^{2t} w_2^2 + \sum_{i=1}^{\ell} \binom{t-i}{i} u_1 v_1^{2t-4i} w_2^{2i+2} + t u_1 w_2^{t+2} \\
&= u_1 w_2 \sum_{i=1}^{2\ell+\kappa} \binom{2t-i}{i} v_1^{2t-2i} w_2^{i+1} + \sum_{i=1}^{\ell} \binom{t-i}{i} u_1 v_1^{2t-4i} w_2^{2i+2} + t u_1 w_2^{t+2}.
\end{aligned}$$

Note that the even indices i in the summation from 1 to $2\ell + \kappa$ cancel out the summation running over $1 \leq i \leq \ell$. On the other hand, if t is odd (i.e. if $\kappa = 1$), then the summand with index $i = 2\ell + 1$ cancels out the final summand $t u_1 w_2^{t+2}$ —if $\kappa = 0$, none of these terms appear. The above expression then simplifies to

$$\rho(d_4 e_m) = u_1 w_2 \sum_{i=1}^{\ell} \binom{2t-2i+1}{2i-1} v_1^{2t-4i+2} w_2^{2i} = \sum_{i=1}^{\ell} \binom{t-i}{i-1} u_1 v_1^{2t-4i+2} w_2^{2i+1}.$$

But (9) implies $u_1 v_1^{2t-4i+2} w_2^{2i+1} = v_1^{2t-4i} \cdot u_1 v_1 \cdot v_1 w_2 \cdot w_2^{2i} = \rho(a_2^{t-2i} b_2 c_3 d_4^i)$ which, as explained in the proof sketch above, gives the d_4 -relation asserted in (14).

References

- [1] D. R. Bausum, “Embeddings and immersions of manifolds in Euclidean space”, *Trans. Amer. Math. Soc.* **213** (1975) 263–303.
- [2] P. V. M. Blagojević and G. M. Ziegler, “Tetrahedra on deformed spheres and integral group cohomology”, *Electron. J. Combin.* **16** No. 2 (2009) Research Paper 16, 11 pp.
- [3] D. M. Davis, “Table of immersions and embeddings of real projective spaces”, available at <http://www.lehigh.edu/~dmd1/immtable>.
- [4] Carlos Domínguez, “Cohomology of pairs of points in real projective spaces and applications”, Ph.D. thesis, Department of Mathematics, Cinvestav, 2011.
- [5] D. Epstein and R. Schwarzenberger, “Imbeddings of real projective spaces”, *Ann. of Math. (2)* **76** (1962) 180–184.
- [6] M. Farber, “Topological complexity of motion planning”, *Discrete Comput. Geom.* **29** (2003) 211–221.

- [7] M. Farber and M. Grant, “Symmetric motion planning”, in *Topology and Robotics, Contemp. Math.* **438**, Amer. Math. Soc., Providence, RI (2007) 85–104.
- [8] M. Farber and M. Grant, “Robot motion planning, weights of cohomology classes, and cohomology operations”, *Proc. Amer. Math. Soc.* **136** (2008) 3339–3349.
- [9] M. Farber, S. Tabachnikov, and S. Yuzvinsky, “Topological robotics: motion planning in projective spaces”, *Int. Math. Res. Not.* **34** (2003) 1853–1870.
- [10] J. González, “Symmetric topological complexity as the first obstruction in Goodwillie’s Euclidean embedding tower for real projective spaces”, to appear in *Trans. Amer. Math. Soc.* (currently available at arXiv:0911.1116v4 [math.AT]).
- [11] J. González and P. Landweber, “Symmetric topological complexity of projective and lens spaces”, *Algebr. Geom. Topol.* **9** (2009) 473–494.
- [12] J. González and P. Landweber, “The integral cohomology groups of configuration spaces of pairs of points in real projective spaces”, initial versions of this paper available at arXiv:1004.0746v1 and arXiv:1004.0746v2.
- [13] D. Handel, “An embedding theorem for real projective spaces”, *Topology* **7** (1968) 125–130.
- [14] D. Handel, “On products in the cohomology of the dihedral groups”, *Tôhoku Math. J. (2)* **45** (1993) 13–42.
- [15] W. Hantzsche, “Einlagerung von Mannigfaltigkeiten in euklidische Räume”, *Math. Zeit.* **43** (1938) 38–58.
- [16] A. Hatcher, *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.
- [17] H. Hopf, “Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume”, *Vierteljschr. Naturforsch. Gesellschaft Zürich* **85** (1940) 165–177.
- [18] S. Kallel, “Symmetric products, duality and homological dimension of configuration spaces”, *Geom. Topol. Monogr.* **13** (2008) 499–527.
- [19] L. L. Larmore and R. D. Rigdon, “Enumerating immersions and embeddings of projective spaces”, *Pacific J. Math.* **64** (1976) 471–492.
- [20] M. Mahowald, “On the embeddability of the real projective spaces”, *Proc. Amer. Math. Soc.* **13** (1962) 763–764.
- [21] M. Mahowald, “On obstruction theory in orientable fibre bundles”, *Trans. Amer. Math. Soc.* **110** (1964) 315–349.

- [22] J. W. Milnor and J. D. Stasheff, *Characteristic Classes*. Annals of Mathematics Studies No. 76, Princeton University Press, Princeton NJ, 1974.
- [23] A. Ranicki, *Algebraic and Geometric Surgery*. Oxford Mathematical Monographs, Oxford Science Publications. Oxford University Press, 2002. Electronic version (August 2009) available at <http://www.maths.ed.ac.uk/~aar/books/surgery.pdf>.
- [24] E. Rees, “Embedding odd torsion manifolds”, *Bull. London Math. Soc.* **3** (1971) 356–362.
- [25] E. Rees, “Embeddings of real projective spaces”, *Topology* **10** (1971) 309–312.

Carlos Domínguez `cda@math.cinvestav.mx`
Departamento de Matemáticas, CINVESTAV-IPN
México City 07000, México

Jesús González `jesus@math.cinvestav.mx`
Departamento de Matemáticas, CINVESTAV-IPN
México City 07000, México

Peter Landweber `landwebe@math.rutgers.edu`
Department of Mathematics, Rutgers University
Piscataway, NJ 08854, USA